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A simple treatment of color screening in a $q\bar{q}$ plasma

G. Calucci[a]

Dipartimento di Fisica teorica dell'Università di Trieste, INFN, Sezione di Trieste, 34014 Trieste, Italy

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Abstract. The formation of Hückel–Debye screening in a $q\bar{q}$ plasma is studied following as close as possible the procedures used in the usual electric case. The equivalent of the Debye length is found, but the screening as a result is found to be less efficient than for the Coulomb interaction, since the correlation functions decay as the exponential of the distance raised at a power less than one.

1 Statement of the problem

The prospect that in a $q\bar{q}$ plasma, produced e.g. in a high energy ion–ion collision, the multi-quark dynamics may build up, and, beyond other more complicated features, something like Hückel–Debye screening occurs, is frequently held, and this is also considered one of the possible indirect signals of the very existence of this state, because it should have some observable effect on the production of the final particles [1,2]. It is not evident that the $q\bar{q}$ plasma be very similar to those simple electromagnetic systems in which the Hückel–Debye screening is produced; however, given some similarities one could investigate whether the prospect may be theoretically fulfilled.

While the theoretical analysis of this effect usually involves the dynamics of the carrier of the interaction [2,3], here the analysis is made in a static frame, keeping the similarity with the electric plasma as close as possible, perhaps even too close, so that one can see how and where the unavoidable differences work. The differences that are here considered, in comparison with the electromagnetic situation are two: one dynamical – the charges are substituted by the color charges and they are non-commuting –, the other statistical – while the system is globally uncolored, it is not truly neutral because realistically there will be more quarks than antiquarks.

The similarities are not only in the statistical description of the system but also in the microscopic dynamics: the interaction due to one gluon exchange and only a static Coulomb force are considered.

At this level of approximation the conclusion is simple: a shielding effect is present also in this case, but it is not purely exponential; the two-body correlations decay more slowly, like $\{\exp[-(ar)^{2/3}]\}/r$ and, moreover, the exponential modulates an oscillating term; the unbalancing of quark and antiquark population seems not to have any relevant consequence.

Since the analogy between the non-commuting and the commuting case plays a relevant role in the treatment, the standard electric case will be summarized in a form particularly suitable for subsequent generalization, and then the treatment of the colored plasma is presented and worked out until the final result is attained. Possible refinements are then discussed and some comparisons with other treatments are outlined. Some of the analytical procedures which have been used are collected in the appendices.

2 Short review of the commutative case

Finding the Hückel–Debye shielding in an assembly of electrically charged particles is a well known text-book [4] procedure; however, it seems useful to present a short summary of the treatment for commuting (electric) charges, in that particular form which is a most convenient starting point for generalization.

One may start from the expression of the canonical partition function

$$Z = \frac{Z_0}{V^N}\int \mathrm{e}^{-\beta U(\mathbf{r})}\mathrm{d}^{3N}r. \tag{1}$$

The interaction term is given by the sum of two-body interactions. We have

$$U = \sum_{i<j} u(|\mathbf{r}_i - \mathbf{r}_j|).$$

The generic term of the sum is the Coulomb interaction $u(r_{i,j}) = \alpha z_i z_j/r$ with $r_{i,j} = |\mathbf{r}_i - \mathbf{r}_j|$, in which the behavior at $r_{i,j} \to 0$ has been suitably regularized in order to prevent a divergence of the partition function. The factor Z_0 takes care of the integration over momenta and of the summation over spin variables.

Then the integrand of the partition function (1) is expanded into multiple correlations as follows:

$$\mathrm{e}^{-\beta U(\mathbf{r})} = \prod_l \mathcal{D}(\mathbf{r}_l) + \sum_{i<j}\mathcal{C}^{(2)}(\mathbf{r}_i,\mathbf{r}_j)\prod_{l\neq i,j}\mathcal{D}(\mathbf{r}_l)$$

[a] e-mail: giorgio@ts.infn.it

$$+\sum_{i<j<k}\mathcal{C}^{(3)}(\mathbf{r}_i,\mathbf{r}_j,\mathbf{r}_k)\prod_{l\neq i,j,k}\mathcal{D}(\mathbf{r}_l)+\dots \quad (2)$$

As already said, the plasma is assumed to be uniform in space, and for this reason the one-particle distribution is a constant, $\mathcal{D}(\mathbf{r}_i)=R$. The functions $\mathcal{C}^{(J)}$ are defined as the pure correlations of order J; they do not embody correlations of lower order so that $\int\mathcal{C}^{(J)}\mathrm{d}^3\mathbf{r}_J=0$; once we have defined R it is convenient to redefine the correlations as $\mathcal{C}^{(J)}=R^JC^{(J)}$. For the two-body distribution we have

$$W(\mathbf{r}_i,\mathbf{r}_j)=\frac{Z_0}{Z}\frac{1}{V^{N-2}}\int \mathrm{e}^{-\beta U(\mathbf{r})}\prod_{l\neq i,j}\mathrm{d}r_l$$
$$=\frac{Z_0}{Z}R^N\left[1+C^{(2)}(\mathbf{r}_i,\mathbf{r}_j)\right]\ . \quad (3)$$

The overall normalization $\int W(\mathbf{r}_i,\mathbf{r}_j)\mathrm{d}^3r_i\mathrm{d}^3r_j/V^2=1$ gives $R=(Z/Z_0)^{1/N}$. So the three-body distribution is

$$W(\mathbf{r}_i,\mathbf{r}_j,\mathbf{r}_k)$$
$$=1+\left[C^{(2)}(\mathbf{r}_i,\mathbf{r}_j)+C^{(2)}(\mathbf{r}_i,\mathbf{r}_k)+C^{(2)}(\mathbf{r}_j,\mathbf{r}_k)\right]$$
$$+C^{(3)}(\mathbf{r}_i,\mathbf{r}_j,\mathbf{r}_k)\ ,$$

and so on.

Now we use (3) and then (2), keeping the dominant terms, i.e. either terms with no $C^{(J)}$ at all or terms with the $C^{(2)}$ factor[1]. We note also that in the integrations in d^3r_l the gradient $\partial u(\mathbf{r}_1,\mathbf{r}_l)/\partial r_{1,v}$ gives zero when integrated alone or multiplied by a term symmetric in $\mathbf{r}_1-\mathbf{r}_l$ and we obtain

$$\frac{\partial C(\mathbf{r}_1,\mathbf{r}_2)}{\partial r_{1,v}}=-\beta$$
$$\times\left(\frac{\partial u(\mathbf{r}_1,\mathbf{r}_2)}{\partial r_{1,v}}+\frac{1}{V}\sum_{l\neq 1,2}\int\mathrm{d}^3r_l\left[\frac{\partial u(\mathbf{r}_1,\mathbf{r}_l)}{\partial r_{1,v}}C(\mathbf{r}_l,\mathbf{r}_2)\right]\right),$$
$$v=x,y,z.$$

Then we take a second derivative so that the Laplace operator acts on C or on u and the equation obtained in doing so yields the final well known result

$$C(r)\propto\frac{1}{r}\exp[-ar],\qquad a=\sqrt{\beta\alpha n},\qquad n=\frac{N}{V}. \quad (4)$$

The length $1/a$ is the Debye radius of the system, and in order for the whole treatment to be consistent we must choose the radius at which we regularize the potential to be much smaller than $1/a$.

[1] For this reason the superscript (2) will be omitted from now on.

3 The non-commutative case

3.1 General features

The starting point is again the expression of the canonical partition function

$$Z=\frac{Z_0}{(3V)^N}\,\mathrm{tr}\int\mathrm{e}^{-\beta U(r)}\mathrm{d}^{3N}r,\quad U=\sum_{i<j}u(|\mathbf{r}_i-\mathbf{r}_j|). \quad (5)$$

The two-body interactions are matrices in the color space and the trace operation acts only on the color indices, which is the reason of the normalizing factor $1/3^N$; flavor variables are not considered. The factor Z_0 takes care of the integration over momenta and of the summation over spin variables.

It is convenient to introduce the variables q_i which embody the space coordinates $\mathbf{r}_i$ and the color indices; thus the integration over q_i indicates both the integration over r_i and the trace over the corresponding color indices.

Then the integrand of the partition function is expanded into multiple correlations as in (1) and the plasma is assumed uniform in space and isotropic in color, so the one-particle distribution is a constant diagonal in the color indices $\mathcal{D}(q_i)=R$. The functions $\mathcal{C}^{(J)}$ are again defined as the pure correlations of order J and we redefine them as $\mathcal{C}^{(J)}=R^JC^{(J)}$. So for the two-body distribution we still have an expansion like (2), with W and C matrices in color space. Now we would like to find an equation for the two-body distribution $W(q_i,q_j)$. In general the derivative of U will not commute with U, because they are matrices, so we use the representation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^A=\int_0^1\mathrm{e}^{xA}\frac{\mathrm{d}A}{\mathrm{d}t}\mathrm{e}^{(1-x)A}\mathrm{d}x, \quad (6)$$

that can be verified by expanding in a series both sides and performing then the integration term by term.

With the use of (2), (3) and (6) it is then possible to write an equation for the two-body correlations $C^{(2)}$. The representation of (6) introduces, through the position $\tau=x\beta$, the distributions at different temperatures, so, when necessary, this dependence will be explicitly written out as follows: $C_\tau(q_i,q_j)$, R_τ.

From (2), by integrating over $\mathrm{d}q_3\dots\mathrm{d}q_N$ and taking again the derivative with respect to $r_{1,v}$, with $v=x,y,z$, we get an equation for the two-body distribution:

$$R_\beta^N\frac{\partial C(q_1,q_2)}{\partial r_{1,v}}$$
$$=-\frac{1}{(3V)^N}\int_0^\beta\mathrm{d}\tau\int\mathrm{d}^{3(N-2)}q\,\mathrm{e}^{-\tau U}\frac{\partial U}{\partial r_{1,v}}\mathrm{e}^{-(\beta-\tau)U}. \quad (7)$$

The trace operation implicit in the q integration acts here over $N-2$ indices. Now (3) is used and when the expansion of (2) is cut at two-body correlations, the result is

$$R_\beta^N \frac{\partial C_\beta(q_1,q_2)}{\partial r_{1,v}} = -\frac{1}{(3V)^{N-2}} \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^{3(N-2)} q R_\tau^N \left[1+\sum_{i<j} C_\tau(q_i,q_j)\right] \times \sum_l \frac{\partial u(q_1,q_l)}{\partial r_{1,v}} R_{\beta-\tau}^N \left[1+\sum_{i<j} C_{\beta-\tau}(q_i,q_j)\right]. \qquad (8)$$

In order to proceed, it is now necessary to get some information about the constants R. By inserting the expansion, (2), into the relation

$$\mathrm{e}^{-\beta U(r)} = \mathrm{e}^{-(\beta-\tau)U(r)} \mathrm{e}^{-\tau U(r)},$$

we get through an integration in $\prod \mathrm{d}q$

$$R_\beta^N = (R_{\beta-\tau}R_\tau)^N \times \left[1+\frac{N(N-1)}{2V^2}\int C_{\beta-\tau}(q_i,q_j)C_\tau(q_i,q_j)\mathrm{d}q_i\mathrm{d}q_j + \dots\right], \qquad (9)$$

where the dots imply higher order or multiple correlations. This result implies that

$$R_\beta = R_{\beta-\tau}R_\tau(1+\Lambda),$$

where Λ is of second (and higher) order in the correlations (see Appendix A). It is clear that most of the integrations are straightforward, and we are left with an integration in $\mathrm{d}^3 r$, a product over a pair of $SU(3)$ indices and an integration over τ. This last integration affects the distributions C but not the potentials $u(q_1,q_k)$. A term with two functions C would give contributions of the same order as a term with one three-body correlation so it is dropped; in the integration over $\mathrm{d}^3 r$ some terms are eliminated by symmetry, precisely as in the electromagnetic case. The final result is then written

$$\frac{\partial C_\beta(q_1,q_2)}{\partial r_{1,v}} = -\int_0^\beta \mathrm{d}\tau \Bigg(\frac{\partial u(q_1,q_2)}{\partial r_{1,v}} + \frac{1}{3V}\sum_{l\neq 1,2}\int \mathrm{d}^3 q_l \left[\frac{\partial u(q_1,q_l)}{\partial r_{1,v}} C_{\beta-\tau}(q_l,q_2) + C_\tau(q_l,q_2)\frac{\partial u(q_1,q_l)}{\partial r_{1,v}}\right]\Bigg). \qquad (10)$$

Now a further derivative with respect to r_1 is performed so that the Laplace operator acts on C and on u; in this last case it gives

$$\Delta u(r_i,r_j) = -\alpha T \delta(\mathbf{r}_i - \mathbf{r}_j).$$

Here $\alpha = g^2/4\pi$ is the QCD coupling, T is the color structure that will be specified. The two-body correlation depends only on the relative distance, and one performs the Fourier transform with respect to it and the Laplace transform with respect to the inverse temperature β with the result

$$-k^2 \check{C}(s;k) = \frac{\alpha T}{s^2} + \frac{\alpha}{3Vs}\left[\sum_{l\neq 1,2} T\check{C}(s;k) + \check{C}(s;k)T\right]. \qquad (11)$$

The check (as in $\check{C}$) means both the Fourier and the Laplace transform; the variable k is the Fourier-conjugated of r and s is the Laplace-conjugated of β.

3.2 Details of the color structure

The factors $\check{C}(s;k)$ and T are matrices so their order is relevant. In order to proceed it is necessary to specify the color structure of the pair of quarks, so the indices will now be displayed[2]. The elementary, Coulomb-like, interaction is not diagonal in the color, so we must distinguish between incoming and outgoing quarks; evidently we must also distinguish between quarks and antiquarks: it is consistent with the group structure to use e.g. lower indices for incoming quarks and outgoing antiquarks and upper indices for incoming antiquarks and outgoing quarks. From the point of view of the color structure the interaction u is a pure octet in the t-channel whereas the correlations C can be a either a singlet or an octet for a $(q\bar{q})$ pair and either an antitriplet or a sextet for a (qq) pair, so we must build up the corresponding projectors for these configurations.

If we have two incoming quarks with color indices a,c and then two outgoing quarks with color indices b,d the color structure of the interaction is[3]

$$I_{a,c}^{b,d} = \tfrac{1}{2}\left[\delta_a^d\delta_c^b - \tfrac{1}{3}\delta_a^b\delta_c^d\right] = \tfrac{1}{4}\sum_{A=1}^{8}(\lambda_A)_a^b(\lambda_A)_c^d,$$

and the projectors for the antitriplet and the sextet are

$$^3\Pi_{a,c}^{b,d} = \tfrac{1}{2}\left[\delta_a^b\delta_c^d - \delta_a^d\delta_c^b\right], \qquad ^6\Pi_{a,c}^{b,d} = \tfrac{1}{2}\left[\delta_a^b\delta_c^d + \delta_a^d\delta_c^b\right].$$

If we have a quark–antiquark pair with incoming color indices a,d and outgoing color indices b,c the interaction has the opposite sign, so its color structure is $-I_{a,c}^{b,d}$, and the projectors for the singlet and the octet are

$$^1\Pi_{a,c}^{b,d} = \tfrac{1}{3}\delta_a^d\delta_c^b, \qquad ^8\Pi_{a,c}^{b,d} = \left[\delta_a^b\delta_c^d - \tfrac{1}{3}\delta_a^d\delta_c^b\right].$$

These projectors are normalized to the multiplicity of the corresponding states:

$$^1\Pi_{f,g}^{f,g} = 1, \qquad ^8\Pi_{f,g}^{f,g} = 8, \qquad ^3\Pi_{f,g}^{f,g} = 3, \qquad ^6\Pi_{f,g}^{f,g} = 6.$$

It is also verified that the interaction is attractive on the singlet and triplet states and repulsive elsewhere:

$$-I_{b,d}^{a,c}\,{}^1\Pi_{a,c}^{b,d} = -4/3, \qquad -I_{b,d}^{a,c}\,{}^8\Pi_{a,c}^{b,d} = 4/3,$$

[2] The first latin letters $a,b,\dots$ will be used as color indices.
[3] For the details of the color representation, see e.g. [5].

$$I^{a,c}_{b,d}\,{}^3\Pi^{b,d}_{a,c} = -2, \qquad I^{a,c}_{b,d}\,{}^6\Pi^{b,d}_{a,c} = 2.$$

As an intermediate step we define the unit tensor $U^{b,d}_{a,c} = \delta^b_a\delta^d_c$; it has the property that $U^{b,d}_{a,c}I^{a,c}_{b,d} = 0$ and

$$^1\Pi = \tfrac{2}{3}I + \tfrac{1}{9}U, \qquad ^8\Pi = -\tfrac{2}{3}I + \tfrac{8}{9}U,$$
$$^3\Pi = -I + \tfrac{1}{3}U, \qquad ^6\Pi = I + \tfrac{2}{3}U.$$

Going back to (11) it is now possible to bring it to a more explicit form: It is necessary to distinguish between quarks and antiquarks both in the correlation functions $\check{C}$ and in the summation $\sum_l$ remembering also that in a real case the quark density ρ will be larger than the antiquark density $\bar{\rho}$. The color of the state appearing in the sum is then summed isotropically.

Defining $\check{C} = M$ for the $(q\bar{q})$ pair, $\check{C} = Q$ for the (qq) pair, and $\check{C} = \bar{Q}$ for the $(\bar{q}\bar{q})$ pair, the equations are

$$\begin{aligned}
&-k^2M^{b,d}_{a,c}\\
&= -\alpha I^{b,d}_{a,c}/s^2 + \alpha\left[\left(Q^{b,g}_{a,f}(-I^{f,d}_{g,c}) + I^{b,g}_{a,f}M^{f,d}_{g,c}\right)\rho\right.\\
&\qquad\left.+\left(M^{b,g}_{a,f}I^{f,d}_{g,c} + (-I^{b,g}_{a,f})\bar{Q}^{f,d}_{g,c}\right)\bar{\rho}\right]/s\,,\\
&-k^2Q^{b,d}_{a,c} \qquad\qquad (12a)\\
&= \alpha I^{b,d}_{a,c}/s^2 + \alpha\left[\left(Q^{b,g}_{a,f}I^{f,d}_{g,c} + I^{b,g}_{a,f}Q^{f,d}_{g,c}\right)\rho\right.\\
&\qquad\left.+\left(M^{b,g}_{a,f}(-I^{f,d}_{g,c}) + (-I^{b,g}_{a,f})M^{f,d}_{g,c}\right)\bar{\rho}\right]/s.
\end{aligned}$$

The equation for the $(\bar{q}\bar{q})$ pair is obtained by the interchange $Q \Leftrightarrow \bar{Q}$ and $\rho \Leftrightarrow \bar{\rho}$ in the equation for the (qq) pair (see Appendix B).

The tensors M, Q, $\bar{Q}$ are then decomposed according to the projectors $^J\Pi$:

$$M = {}^1\Pi F_1 + {}^8\Pi F_8, \quad Q = {}^3\Pi F_3 + {}^6\Pi F_6,$$
$$\bar{Q} = {}^3\Pi\bar{F}_3 + {}^6\Pi\bar{F}_6.$$

Then one uses the relations, that are easily verified,

$$^1\Pi^{b,g}_{a,f}I^{f,d}_{g,c} = \tfrac{1}{3}I^{b,d}_{a,c}, \qquad ^8\Pi^{b,g}_{a,f}I^{f,d}_{g,c} = -\tfrac{1}{3}I^{b,d}_{a,c},$$
$$^3\Pi^{b,g}_{a,f}I^{f,d}_{g,c} = -\tfrac{1}{2}I^{b,d}_{a,c}, \qquad ^6\Pi^{b,g}_{a,f}I^{f,d}_{g,c} = \tfrac{1}{2}I^{b,d}_{a,c}.$$

So the RHS of (12a) contains only the $I^{b,d}_{a,c}$ tensor, and by projecting out the $U^{b,d}_{a,c}$ term from the LHS we get the identities

$$F_8 = -\tfrac{1}{8}F_1, \quad F_6 = -\tfrac{1}{2}F_3, \quad \bar{F}_6 = -\tfrac{1}{2}\bar{F}_3\,.$$

In this way (12) are reduced to

$$\begin{aligned}
&k^2F_1 - \tfrac{4}{3}\alpha/s^2 + \alpha\left[\rho F_3 + \bar{\rho}\bar{F}_3 + \tfrac{1}{2}(\rho+\bar{\rho})F_1\right]/s = 0,\\
&k^2F_3 - \tfrac{2}{3}\alpha/s^2 + \alpha\left[\rho F_3 + \tfrac{1}{2}\bar{\rho}F_1\right]/s = 0,\\
&k^2\bar{F}_3 - \tfrac{2}{3}\alpha/s^2 + \alpha\left[\bar{\rho}\bar{F}_3 + \tfrac{1}{2}\rho F_1\right]/s = 0. \qquad (12b)
\end{aligned}$$

This system yields immediately $F_1 = F_3 + \bar{F}_3$ and is then reduced to a two-equation system:

$$\begin{aligned}
&k^2F_3 - \tfrac{2}{3}\alpha/s^2 + \alpha\left[\left(\rho + \tfrac{1}{2}\bar{\rho}\right)F_3 + \tfrac{1}{2}\bar{\rho}\bar{F}_3\right]/s = 0,\\
&k^2\bar{F}_3 - \tfrac{2}{3}\alpha/s^2 + \alpha\left[\left(\bar{\rho} + \tfrac{1}{2}\rho\right)\bar{F}_3 + \tfrac{1}{2}\rho F_3\right]/s = 0. \qquad (12c)
\end{aligned}$$

The solution depends only on the total fermionic density $n = \rho + \bar{\rho}$ and is

$$F_3 = \bar{F}_3 = \tfrac{2}{3}\alpha\frac{1}{k^2s^2 + n\alpha s} = \check{G}(k^2, s). \qquad (13)$$

From

$$\check{G}(k^2, s) = \frac{2}{3n}\left[\frac{1}{s} - \frac{1}{s + \alpha n/k^2}\right]$$

we get its Laplace anti-transform:

$$\hat{G}_\beta(k^2) = \frac{2}{3n}\left[1 - \exp\left[-\beta\alpha n/k^2\right]\right]. \qquad (14)$$

From this expression it is possible to calculate the correlation energy, but in order to understand how the correlations behave in space the Fourier transform is needed:

$$G_\beta(r^2) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot r}\hat{G}_\beta(k^2)d^3k. \qquad (15)$$

After performing the angular integration the resulting expression is estimated by means of the saddle point method, for large values of r:

$$G_\beta(r^2) = \frac{1}{2\pi^2 r}\Im\int_0^\infty e^{ikr}\hat{G}_\beta(k^2)k\,dk,$$

with the result (see Appendix C)

$$\begin{aligned}
G_\beta(r^2) &\propto \frac{1}{r}\exp\left[-\tfrac{3}{2}(ar/2)^{2/3}\right] \qquad (16)\\
&\quad\times\cos\left[\tfrac{3}{2}\sqrt{3}(ar/2)^{2/3} - \tfrac{1}{3}\pi\right] + \ldots
\end{aligned}$$

The subsequent terms contain higher negative powers of r but the same exponential behavior.

The result expressed in (16) is quite simple: it says that the Debye shielding is present also in this case, but there are two differences. The shielding is not a simple exponential, the decay is in fact slower since at the exponent on find a power of r smaller than 1, and moreover there is an oscillating behavior, controlled by the same parameter a. We note also that in this approximation the behavior of the $(q\bar{q})$ pair is the same as the behavior of the (qq) pair.

4 Possible developments and conclusions

The result expressed in (16) comes from particular simplifications that have been used, following the idea of reproducing what is done in the commutative case with a minimum of further complications.

A lot of refinements could be foreseen, but it is not obvious that they are relevant for the present level of knowledge of the QCD plasma.

We may start from the consideration of the possible corrections to (10). It is not difficult to correct the density term R, see (9) and Appendix A. To be consistent we should include also quadratic terms in $C^{(2)}$ in (11) and this is certainly more complicated, but if we consider twice the term $C^{(2)}$ we should consider also the three-body correlation $C^{(3)}$, so we see that the whole situation rapidly becomes cumbersome[4]. Luckily we have seen that the two-body correlations are enough to give an answer in the commutative case and also to give a definite, different answer in the non-commutative case.

Another point is the inclusion of the gluons in the dynamics: we can foresee at least three effects.
(1) The virtual gluons give rise (together with the virtual quarks) to the presence of a running coupling constant in (14). Until a perturbative treatment holds the variation of α is not very strong; it is in fact logarithmic in k^2, so the further application of the stationary phase in estimating the Fourier transform, although very complicated analytically, will not give a result very different from (16).
(2) The inclusion of real gluons present in the plasma will modify (12) and the subsequent ones. In order to close the system we must include also the two-gluon state, which is certainly not interesting phenomenologically, but is dynamically coupled with the (qq) and $(q\bar{q})$ states. Perhaps this is the most important of the neglected effects; it has no obvious commutative counterpart.
(3) A further effect comes from the exchange of two gluons in the t-channel. In this case we could have also a color-singlet exchange and some of the relations that allow the simplification leading to (12b) and (12c) are lost, but this is certainly a higher order effect.

This first investigation gives anyhow a reasonably transparent answer; it is what was expected, but not in a trivial way. The shielding parameter a is found to be the same as in the commutative case, but for the numerical factor $\sqrt{27/32}$ (see (16)). This is not strange. As far as the shielding effect exists the structure of a is dictated by dimensional requirements; the shape of the shielding is definitely different from the commutative case; it is slightly broader, and there is no evident a priori reason for this result.

Some observation can however be made. A falling off slower than exponential could be obtained by means of a superposition of Yukawa functions:

$$\tilde{G} \propto \frac{1}{r} \int \sigma(\mu) \mathrm{e}^{-\mu r} \mathrm{d}\mu,$$

provided the density σ remains positive as $\mu \to 0$; it must however go to zero faster than any power in order to reproduce a damping which is also faster than any power. In particular one verifies that if the spectral density $\sigma(\mu)$ behaves as $\exp[-C/\mu^2]$ a behavior like $\exp[-(ar)^{2/3}]$ is generated. Different kinds of investigations have been performed of the form of the spectral density $\sigma(\mu)$. There are analyses, performed, from the beginning, with general non-perturbative procedures [3,7,8] using strong coupling methods. In these cases more than the effects of shielding in the interaction of two color charges due to the presence of other color the effects leading ultimately to the confinement are put into evidence, so a Yukawa fall off is a natural outcome. The treatment presented here is a weak coupling calculation, just in the same sense as the standard Hückel–Debye is a weak coupling effect in QED; a possible concrete application is foreseen in a situation where there is a plasma with a spatial extension larger than the hadron radius so that there is room enough to find a shielding effect in its interior. Obviously at its borders the color is anyhow confined and the gluon propagator will ultimately show a spatial decay of the Yukawa type if not faster.

The result is the same for the (qq) and $(q\bar{q})$ states; there only one density parameter, given by the sum of the quark and antiquark density; however, the treatment emphasizes the role of the quark with respect to the role of the gluons so it should be more appropriate for situations with a large quark density[5]. The (qq) system is not interesting by itself, but because the formation of a baryon implies certainly three (qq) bindings and if the shielding acts in the sense of making the binding less effective, we should expect effects of the same size for mesons and for baryons; of course a genuine three-body effect could be present in the baryon, but this is beyond the scope of the present investigation.

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Appendix A

Although this result is not used furthermore here it is shown how to calculate the corrections to the density parameter R.

If we cut (9) at zero order, then we should conclude that $R_\beta = \mathrm{e}^{\beta\mathcal{F}}$, with $\mathcal{F}$ still undetermined, but not relevant for the subsequent calculations, and we see that since $R = (Z/Z_0)^{1/N}$, this quantity, $\mathcal{F}$, is a sort of constant shift of the free energy per particle.

Now we define $R_\beta = \mathrm{e}^{\beta\mathcal{F}}(1+\delta_\beta)$ and we get from (9), at first order in the correction,

$$1+\delta_\beta = 1 + \delta_{\beta-\tau} + \delta_\tau + \left[1 + \frac{N}{2V^2}\int C_{\beta-\tau}(q_i,q_j)C_\tau(q_i,q_j)\mathrm{d}q_i\mathrm{d}q_j + \ldots\right].$$

One integral is trivial and gives V, the other is calculated by using the k representation for the correlation functions, (15), with the result

$$\frac{1}{(2\pi)^3}\,\mathrm{tr}\int \mathrm{d}^3k\hat{G}_{\beta-\tau}(k^2)\hat{G}_\tau(k^2)$$

[4] The extension of the formulation that has been used here to the case of higher order correlation may not be trivial [6], even in the commutative case.

[5] A similar system was considered e.g. by Csörgő. [9]

$$= \frac{4}{3\sqrt{n}} \left(\frac{\alpha}{\pi}\right)^{3/2} \left[\beta^{3/2} - (\beta-\tau)^{3/2} - \tau^{3/2}\right] ;$$

note that a factor 9 comes from the trace; and we get the explicit expression

$$\delta_\beta = \frac{2\sqrt{n}}{3} \left(\frac{\alpha\beta}{\pi}\right)^{3/2} .$$

Appendix B

In this short appendix a detailed example of the construction of the interaction term as in (12) is presented for what concerns the color variables. We have an incoming quark a and an incoming antiquark d, and they may interact directly becoming an outgoing quark c and an outgoing antiquark b. This is the term $-I^{b,d}_{a,c}$, or the quark a together with another quark f builds up a di-quark system: $Q^{b,g}_{a,f}$, and this partner quark interacts then with the antiquark. This is the term $-I^{f,d}_{g,c}$. We could also consider the interaction of the initial quark with the new-coming one: this is the term $-I^{b,g}_{a,f}$ when this third quark forms a $(q\bar{q})$-system with the original antiquark.

All the other terms are variations of this procedure; these variations in particular let us foresee the possibility for the third particle to be an antiquark. The fact that the indices of the third quark appear always as dummy reflects the fact that we take a symmetric mean value, which is justified only if there are many quarks and antiquarks around.

Appendix C

Here the saddle point estimate [10] of the large r behavior of the correlation is exposed in detail.

We start from (15):

$$G_\beta(r^2) = \frac{1}{(2\pi)^3} \int e^{i\mathbf{k}\cdot r} \hat{G}_\beta(k^2) d^3k.$$

The angular integration is standard and yields the expression

$$G_\beta(r^2) = \frac{1}{2\pi^2 r} \Im \int_0^\infty e^{ikr} \hat{G}_\beta(k^2) k dk$$
$$= -\frac{1}{2\pi^2 r} \Delta_r \Im \int_0^\infty e^{ikr} \frac{2}{3n} \left[1 - \exp\left[-\beta\alpha n/k^2\right]\right] dk/k.$$

The first addendum in the brackets is integrated explicitly and gives a result independent of r, so it drops out in the derivative. The second is written as

$$K \int e^{i\phi} dk/k, \quad \text{where} \quad \phi = kr + i\beta\alpha n/k^2;$$

then one sets $a^2 = \beta\alpha n$ and $k = a^{2/3} r^{-1/3} p$ with the result $\phi = (ar)^{2/3}(p + i/p^2)$. The stationary phase is found at the three complex points $p = \sqrt[3]{2}\left[-i, \frac{1}{2}(-\sqrt{3}+i), \frac{1}{2}(\sqrt{3}+i)\right]$ but only the third one lies in the positive half plane; note that the integration path must be tangent to the positive real axis at the origin since there an essential singularity is present, which is integrable only with this choice. So the phase is expanded around $p_0 = \frac{1}{2}(\sqrt{3}+i)$, with the result

$$i\phi \approx \tfrac{3}{2}(ar)^{3/2} \left[\sqrt[3]{2} e^{2\pi i/3} - \tfrac{1}{2}\sqrt[3]{4} e^{-2\pi i/3}(p-p_0)^2\right].$$

The final Gaussian interaction is performed with the substitution $p = p_0 + s e^{2i\pi/3}$ and the result, which contains a number of non-interesting numerical factors, is

$$\int e^{i\phi} dk/k \approx \sqrt{\frac{4\pi}{3}}\, i \frac{1}{\sqrt[3]{2ar}} \exp\left[-\tfrac{3}{2}(ar/2)^{2/3}(1 - i\sqrt{3})\right].$$

We must now apply the Laplace operator and then take the imaginary part. A behavior like

$$G_\beta(r^2) \propto \frac{1}{r} \exp\left[-\tfrac{3}{2}(ar/2)^{2/3}\right] \times \cos\left[\tfrac{3}{2}\sqrt{3}(ar/2)^{2/3} - \tfrac{1}{3}\pi\right] + \ldots$$

is obtained, where the dots represent terms with higher negative powers, at least a factor $r^{-3/2}$ more, but the same exponential behavior.

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